An Introduction to Integral Equations

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ICERM Workshop on

Fast Algorithms for Generating Static and Dynamically Changing Point Configurations

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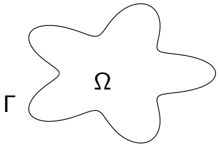
Linear boundary value problems

We consider a Poisson problem
Dirichlet boundary condition:



$$\begin{cases} -\Delta\, u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \quad\; u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

However, the solution techniques can be extended to linear boundary value problems of the form

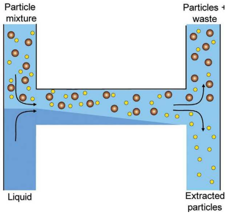
$$\begin{cases} A\, u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \Omega, \\ B\, u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Gamma, \end{cases} \qquad \text{(BVP)}$$

where Ω is a domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with boundary Γ. For instance:

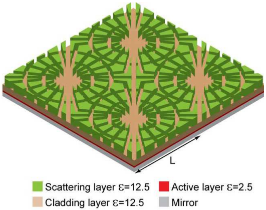
- The equations of linear elasticity.
- Stokes' equation.
- Helmholtz' equation (at least at low and intermediate frequencies).
- Time-harmonic Maxwell (at least at low and intermediate frequencies).

Example applications

(a)



(b)



(c)

(a) The Wraith Virginia-class submarine

(b) Kayani, A., Khoshmanesh, K., Ward, S., Mitchell, A., and Kalantar-zadeh, K. Optofluidics incorporating actively controlled micro- and nano-particles. In Biomicrofluidics, vol. 6.

Boundary value problem

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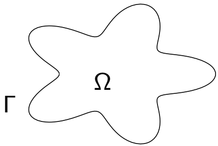

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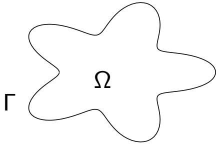

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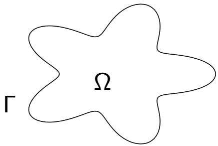


$$\begin{cases} -\Delta\, u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \quad\; u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

Let's write $u(\mathbf{x}) = v(\mathbf{x}) + w(\mathbf{x})$ where $v(\mathbf{x})$ is the solution of

$$-\Delta\, v(\mathbf{x}) = \hat{g}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,$$

and $w(\mathbf{x})$ is solution of

$$\begin{cases} -\Delta\, w(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ \quad\; w(\mathbf{x}) = f(\mathbf{x}) - v(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

The function $v(\mathbf{x})$ is called the particular solution and $w(\mathbf{x})$ is called the homogeneous solution.

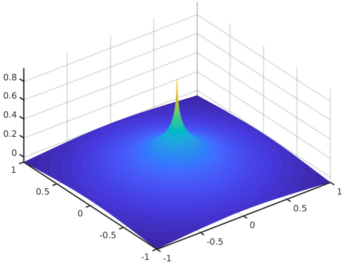
The fundamental solution

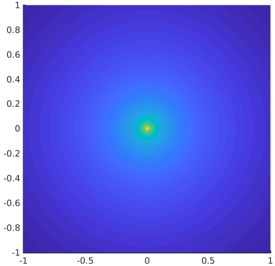
For a given point charge $\mathbf{x}_0 \in \mathbb{R}^2$, the solution of

$$-\Delta u(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0), \quad \mathbf{x} \in \mathbb{R}^2$$

is

$$u(\mathbf{x}) = -\frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0|.$$





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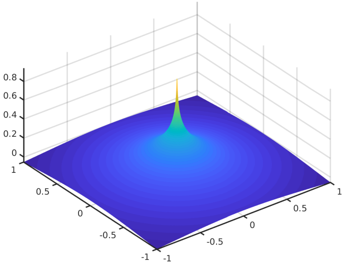
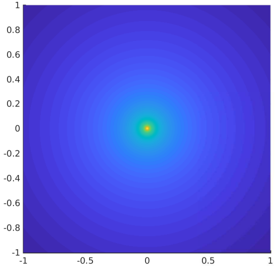
$$u(\mathbf{x}) = -\frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}_0|.$$

The fundamental solution $G(\mathbf{x}, \mathbf{y})$ is given by

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log |\mathbf{x} - \mathbf{y}|.$$

This allows us to move the point charge around.

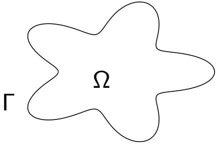




The particular solution

Recall, $v(\mathbf{x})$ satisfies

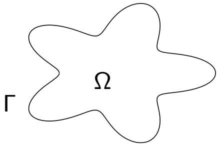
$$-\Delta\, v(\mathbf{x}) = \hat{g}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,$$



where

$$\hat{g}(\mathbf{x}) = \begin{cases} g(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega \\ 0 & \text{for } \mathbf{x} \in \Omega^c \end{cases}$$

The particular solution



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Using the fundamental solution, the particular solution is given by

$$v(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{y}) g(\mathbf{y}) dA(\mathbf{y}).$$

The homogeneous solution

Recall, $w(\mathbf{x})$ is the solution to the boundary value problem with homogeneous partial differential equation; i.e.

$$\begin{cases} -\Delta\, w(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ \quad w(\mathbf{x}) = f(\mathbf{x}) - v(\mathbf{x}) = \hat{f}(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

It is tempting to express $w(\mathbf{x})$ as

$$w(\mathbf{x}) = \int_\Gamma G(\mathbf{x}, \mathbf{y})\sigma(\mathbf{y})dl(\mathbf{y})$$

where $\sigma(\mathbf{y})$ is an unknown boundary charge distribution.

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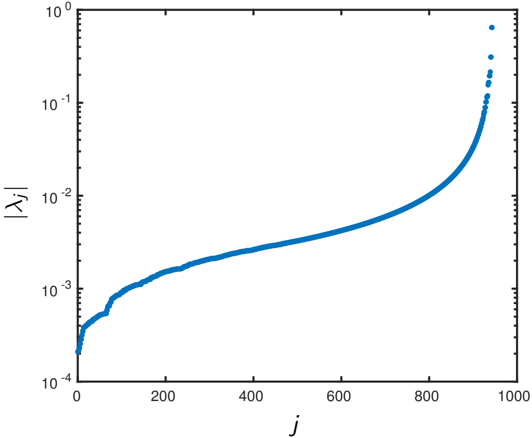
Enforcing the boundary condition yields the following first kind Fredholm equation

$$\int_\Gamma G(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})dl(\mathbf{y}) = \hat{f}(\mathbf{x}) \quad \mathbf{x} \in \Gamma.$$

The spectrum



The minimum eigenvalue in absolute value is $2.06e-04$.
The maximum eigenvalue in absolute value is $6.39e-1$.

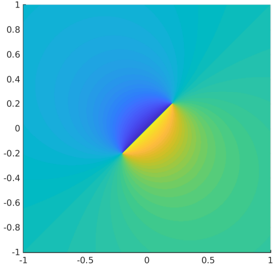
The double layer kernel

For a point $\mathbf{y}$ on the boundary of a curve, the double layer kernel

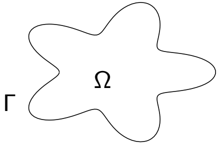
$$D(\mathbf{x},\mathbf{y}) = \partial_{\nu_\mathbf{y}} G(\mathbf{x},\mathbf{y})$$

is a solution of

$$-\Delta_\mathbf{x} w(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{y}), \qquad \mathbf{x}\in\mathbb{R}^2.$$



A second kind integral equation



Consider the problem

$$
\begin{aligned}
-\Delta w(\mathbf{x}) &= 0, \qquad \mathbf{x} \in \Omega, \\
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\end{aligned}
$$

The solution can be represented as a double layer potential

$$s(\mathbf{x}) = \int_\Gamma \partial_{\boldsymbol{\nu}_\mathbf{y}} G(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})ds(\mathbf{y}), \qquad \mathbf{x} \in \Omega,$$

where $\boldsymbol{\nu}_\mathbf{y}$ is the outward normal at $\mathbf{y}$ and $G(\mathbf{x},\mathbf{y})$ is the fundamental solution

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|.$$

Then the boundary charge distribution σ satisfies the boundary integral equation

$$\frac{1}{2}\sigma(\mathbf{x}) + \int_\Gamma \partial_{\boldsymbol{\nu}_\mathbf{y}} G(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})ds(\mathbf{y}) = \hat{f}(\mathbf{x})$$

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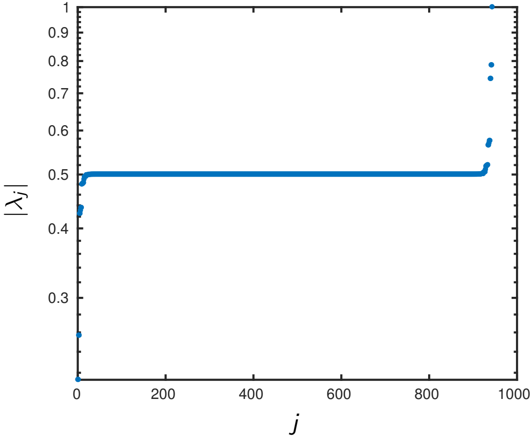
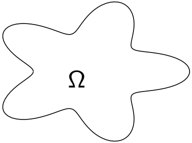


Variable coefficient PDEs



Consider the free space variable coefficient Poisson problem

$$\nabla \cdot (a(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{R}^2$$

where $a(\mathbf{x}) > 0$ for $\mathbf{x} \in \Omega$ and the support of $f(\mathbf{x})$ is Ω.

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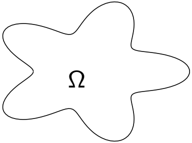

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Expanding the differential operator (plus some algebra) results in the following form of the PDE;

$$\Delta u(\mathbf{x}) + \frac{\nabla a(\mathbf{x}) \cdot \nabla u(\mathbf{x})}{a(\mathbf{x})} = \frac{f(\mathbf{x})}{a(\mathbf{x})} \quad \text{for } \mathbf{x} \in \mathbb{R}^2.$$

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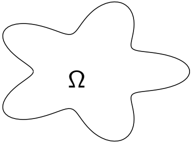

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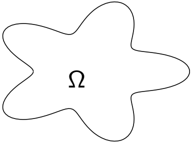
We let $u(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})dA(\mathbf{y})$ and plug this expression into the PDE.

$$\sigma(\mathbf{x}) + \int_\Omega \frac{\nabla a(\mathbf{x}) \cdot (\nabla_x G(\mathbf{x},\mathbf{y}))}{a(\mathbf{x})}\sigma(\mathbf{y})dA(\mathbf{y}) = \frac{f(\mathbf{x})}{a(\mathbf{x})} \quad \text{for } \mathbf{x} \in \mathbb{R}^2.$$

Variable coefficient PDEs



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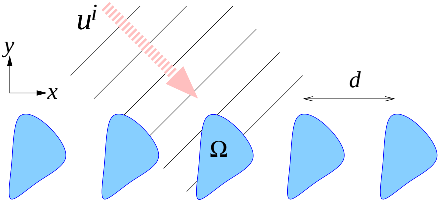
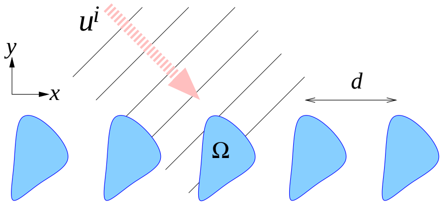
Definition of quasi-periodic scattering



- Let $\Omega \subset \mathbb{R}^2$ denote one obstacle. Then the collection of obstacles is expressed as $\Omega_{\mathbb{Z}} = \{\mathbf{x} : (x + nd, y) \in \Omega \text{ for some } n \in \mathbb{Z}\}$.
- The obstacles are hit by an incident plane wave $u^{\text{inc}} = e^{i\mathbf{k}\cdot\mathbf{x}}$ where $|k| = \omega$.
- Our goal is to find the total field $u^{\text{total}} = u^{\text{inc}} + u$.
- Utilize the fact that each part of the field is quasi-periodic: ie. $u(x + d, y) = \alpha u(x, y)$ where $\alpha = e^{i\kappa^i d}$ denotes the Bloch phase.

Differential equation

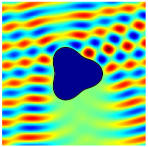


$$(\Delta + \omega^2)u(\mathbf{x}) = 0 \qquad \mathbf{x} \in \mathbb{R}^2 \setminus \Omega_{\mathbb{Z}}$$

$$u(\mathbf{x}) = -u^{\text{inc}}(\mathbf{x}) \qquad \mathbf{x} \in \partial\Omega_{\mathbb{Z}}$$

$$u \quad \text{`radiative'} \text{ as } \ \text{y} \to \pm\infty$$

Single body scattering



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$$u \quad \text{'radiative' far from } \Omega$$

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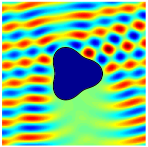

Consider the problem

$$\begin{aligned} &(\Delta + \omega^2)u(\mathbf{x}) = 0 \qquad && \mathbf{x} \in \mathbb{R}^2 \setminus \Omega \\ &u(\mathbf{x}) = -u^{\text{inc}}(\mathbf{x}) && \mathbf{x} \in \partial\Omega \\ &u \quad \text{'radiative' far from } \Omega \end{aligned}$$

The solution can be represented as a double layer potential

$$u(\mathbf{x}) = \int_{\partial\Omega} \partial_{\boldsymbol{\nu}} G_\omega(\mathbf{x}, \mathbf{y})\tau(\mathbf{y})ds(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R} \setminus \Omega,$$

where $\boldsymbol{\nu}$ is the outward normal and $G_\omega(\mathbf{x}, \mathbf{y})$ is the fundamental solution

$$G_\omega(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(\omega|\mathbf{x} - \mathbf{y}|).$$

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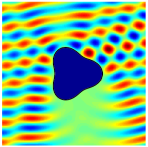

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\begin{array}{ll}
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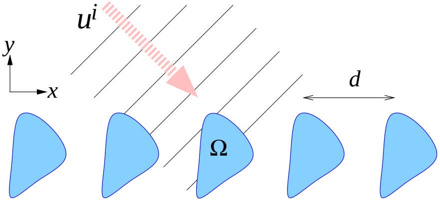
$$G_\omega(\mathbf{x},\mathbf{y})=\frac{i}{4}H_0^{(1)}(\omega|\mathbf{x}-\mathbf{y}|).$$

Then the boundary charge distribution τ satisfies the boundary integral equation

$$\frac{1}{2}\tau(\mathbf{x})+\int_{\partial\Omega}\partial_{\boldsymbol{\nu}}G_\omega(\mathbf{x},\mathbf{y})\tau(\mathbf{y})ds(\mathbf{y})=-u^{\mathrm{inc}}(\mathbf{x})$$

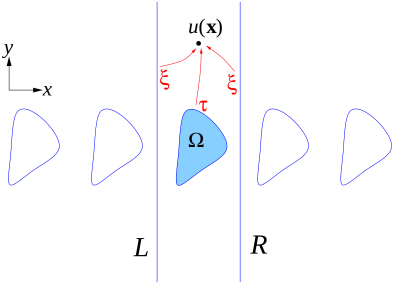
The standard way



Use same integral equation as before but replace $G_\omega(\mathbf{x}, \mathbf{y})$ by $G_{\omega,\mathrm{QP}}(\mathbf{x}) := \sum_{m \in \mathbb{Z}} \alpha^m G_\omega(\mathbf{x} - md)$ where α is the Bloch phase.

This has some problems...

One approach to solving the periodic problem



Let the solution be represented as a double layer potential plus a quasi-periodic potential

$$u(\mathbf{x}) = \int_{\partial\Omega} \partial_{\boldsymbol{\nu}} G_\omega(\mathbf{x},\mathbf{y})\tau(\mathbf{y})ds(\mathbf{y}), + u_{QP}[\xi].$$

New condition: vanishing 'discrepancy'

$$\begin{cases} u_L - \alpha^{-1}u_R &= 0 \\ u_{nL} - \alpha^{-1}u_{nR} &= 0 \end{cases}$$

L. Greengard and A. Barnett (2011)

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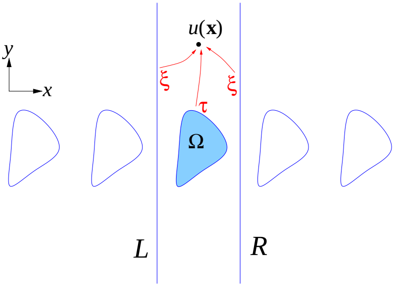


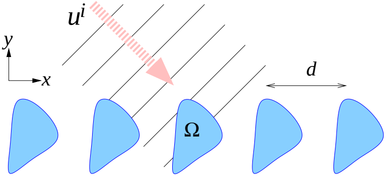
The discretization of the resulting integral equations leads to the following $(N+M)\times(N+M)$ linear system

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \tau \\ \xi \end{bmatrix} = \begin{bmatrix} -\mathbf{u}^{\text{inc}} \\ \mathbf{0} \end{bmatrix},$$

where **A** results from the discretization of the double layer kernel, **B**, **C**, and **Q** are used to enforce the new boundary conditions.

L. Greengard and A. Barnett (2011)

A fast quasi periodic solver



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Instead of directly inverting the matrix, we can compute the solution via a 2×2 block solve.

$$\xi = (\mathbf{Q} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{A}^{-1}\mathbf{u}^{\mathrm{inc}}$$

$$\tau = \mathbf{A}^{-1}\mathbf{u}^{\mathrm{inc}} - \mathbf{A}^{-1}\mathbf{B}\xi$$

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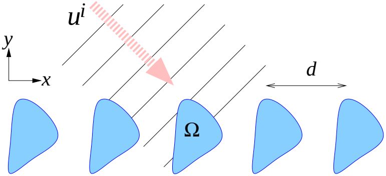


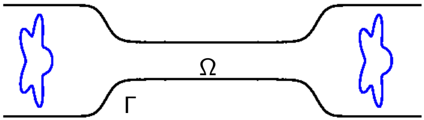
$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \tau \\ \xi \end{bmatrix} = \begin{bmatrix} -\mathbf{u}^{\text{inc}} \\ \mathbf{0} \end{bmatrix}$$

Instead of directly inverting the matrix, we can compute the solution via a 2×2 block solve.

$$\xi = (\mathbf{Q} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{A}^{-1}\mathbf{u}^{\text{inc}}$$
$$\tau = \mathbf{A}^{-1}\mathbf{u}^{\text{inc}} - \mathbf{A}^{-1}\mathbf{B}\xi$$

Note: $\mathbf{A}^{-1}$ need only be computed once independent of the number of incident angles.

Problem statement

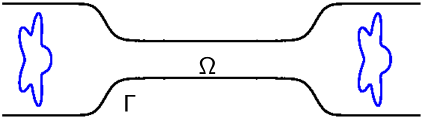


$$
\begin{aligned}
-\mu\Delta\mathbf{u} + \nabla\mathbf{p} &= \mathbf{f}_{ves} & \mathbf{x} \in \Omega\\
\nabla\cdot\mathbf{u} &= 0 & \mathbf{x} \in \Omega\\
\mathbf{u} &= 0 & \mathbf{x} \in \Gamma
\end{aligned}
$$

where **p** denotes the pressure and μ denotes the viscosity.

Vesicle movie

Classic periodizing approach



The velocity field can be expressed as

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= [\mathcal{D}^P_U \boldsymbol{\tau}_U](x) - [\mathcal{D}^P_D \boldsymbol{\tau}_D](x) + \mathcal{S}\mathbf{f}_{ves}\\ &= \int_U D^P(\mathbf{x},\mathbf{y})\boldsymbol{\tau}_U(\mathbf{y})ds_{\mathbf{y}} + \int_D D^p(\mathbf{x},\mathbf{y})\boldsymbol{\tau}_D(\mathbf{y})ds_{\mathbf{y}} + \mathcal{S}\mathbf{f}_{ves}\end{aligned}$$

where

$$D^p = \sum_{n\in\mathbb{Z}} D(\mathbf{x},\mathbf{y}+n\mathbf{d}), \quad d_1 = p,\ d_2 = 0,$$

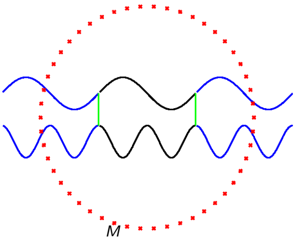
and p is the period of the flow.

Then $\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\tau}_U \\ \boldsymbol{\tau}_D \end{bmatrix}$ satisfies the following integral equation

$$\left(-\frac{1}{2}\mathcal{I} + \mathcal{D}^P\right)\boldsymbol{\tau} = \mathbf{f}$$

for $\mathbf{f} = -\begin{bmatrix} \mathcal{S}\mathbf{f}_{ves,U} \\ \mathcal{S}\mathbf{f}_{ves,D} \end{bmatrix}$.

Periodizing scheme



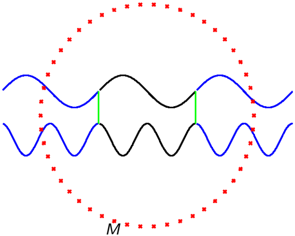
We make the ansatz that the velocity can be represented by

$$\mathbf{u}(\mathbf{x}) = [\mathcal{D}^l_U \boldsymbol{\tau}_U](\mathbf{x}) + [\mathcal{D}^l_D \boldsymbol{\tau}_D](\mathbf{x}) + \mathcal{S}\mathbf{f}_{ves} + \sum_{m=1}^{M} c_m \phi_m(\mathbf{x})$$

where $\phi_m = \mathcal{S}(\mathbf{x}, \mathbf{y}_m)$ a Stokeslets charge, $\{\mathbf{y}_m\}_{m=1}^M$ are a collection of proxy points, and $\mathcal{D}^l$ denotes the local copies of $\mathcal{D}$ given by

$$[\mathcal{D}^l \boldsymbol{\tau}](\mathbf{x}) = \sum_{n=-1}^{1} \mathcal{D}(\mathbf{x}, \mathbf{y} + n\mathbf{d}).$$

Periodizing scheme



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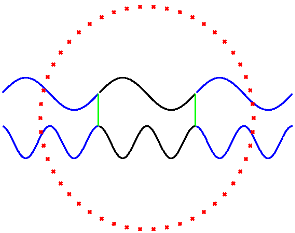
$$[\mathcal{D}^l \boldsymbol{\tau}](\mathbf{x}) = \sum_{n=-1}^{1} \mathcal{D}(\mathbf{x}, \mathbf{y} + n\mathbf{d}).$$

To enforce the periodicity of the solution, we require

$$\mathbf{u}_L - \mathbf{u}_R = 0$$
$$\mathbf{T}(\mathbf{u}, \mathbf{p})_L - \mathbf{T}(\mathbf{u}, \mathbf{p})_R = 0$$

where $\mathbf{T}$ denotes the traction operator.

Periodizing scheme



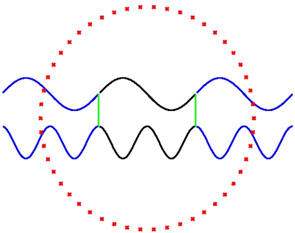
We make the ansatz that the velocity can be represented by
$\mathbf{u}(\mathbf{x}) = [\mathcal{D}^l_U \boldsymbol{\tau}_U](\mathbf{x}) + [\mathcal{D}^l_D \boldsymbol{\tau}_D](\mathbf{x}) + \mathcal{S}\mathbf{f}_{ves}$
$+\sum_{m=1}^M c_m \phi_m(\mathbf{x})$.

Enforcing the no slip boundary condition on the upper and lower surfaces results in the following integral equations

$$(-\frac{1}{2}\mathcal{I} + \mathcal{D}^l_{UU})\boldsymbol{\tau}_U + \mathcal{D}^l_{UD}\boldsymbol{\tau}_D + \sum_{m=1}^{M} c_m \phi_m|_U = -\mathcal{S}\mathbf{f}_{ves}|_U$$

$$\mathcal{D}^l_{DU}\boldsymbol{\tau}_U + (-\frac{1}{2}\mathcal{I} + \mathcal{D}^l_{DD})\boldsymbol{\tau}_D + \sum_{m=1}^{M} c_m \phi_m|_U = -\mathcal{S}\mathbf{f}_{ves}|_D$$

Periodizing scheme



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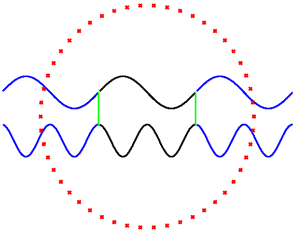
In block matrix form,

$$[\mathbf{A} \quad \mathbf{B}] \begin{bmatrix} \boldsymbol{\tau} \\ \mathbf{c} \end{bmatrix} = \mathbf{f}.$$

Periodizing scheme



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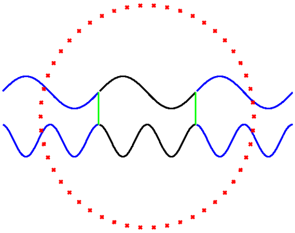
Enforcing the periodicity conditions (and after lots of cancellation) yields the following integral equations

$$(\mathcal{D}_{LU}^{+1} - \mathcal{D}_{RU}^{-1})\boldsymbol{\tau}_U + (\mathcal{D}_{LD}^{+1} - \mathcal{D}_{RD}^{-1})\boldsymbol{\tau}_D + \sum_{m=1}^{M} c_m(\phi_{m,L} - \phi_{m,R}) = -\left(\mathcal{S}_{L,ves}^{+1} - \mathcal{S}_{R,ves}^{-1}\right)\mathbf{f}_{ves}$$

$$(\mathcal{T}_{LU}^{+1} - \mathcal{T}_{RU}^{-1})\boldsymbol{\tau}_U + (\mathcal{T}_{LD}^{+1} - \mathcal{T}_{RD}^{-1})\boldsymbol{\tau}_D + \sum_{m=1}^{M} c_m(\phi_{m,L} - \phi_{m,R})_T = -\left(\mathcal{D}_{L,ves}^{+1} - \mathcal{D}_{R,ves}^{-1}\right)\mathbf{f}_{ves}$$

Periodizing scheme



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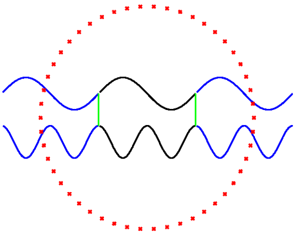
$$(\mathcal{T}^{+1}_{LU} - \mathcal{T}^{-1}_{RU})\boldsymbol{\tau}_U + (\mathcal{T}^{+1}_{LD} - \mathcal{T}^{-1}_{RD})\boldsymbol{\tau}_D + \sum_{m=1}^{M} c_m(\phi_{m,L} - \phi_{m,R})_T = -\left(\mathcal{D}^{+1}_{L,ves} - \mathcal{D}^{-1}_{R,ves}\right)\mathbf{f}_{ves}$$

In block matrix form,

$$[\mathbf{C} \quad \mathbf{Q}]\begin{bmatrix}\boldsymbol{\tau}\\ \mathbf{c}\end{bmatrix} = \mathbf{g}.$$

Periodizing scheme



We make the ansatz that the velocity can be represented by

$$\mathbf{u}(\mathbf{x}) = [\mathcal{D}^{l}_{U}\boldsymbol{\tau}_{U}](\mathbf{x}) + [\mathcal{D}^{l}_{D}\boldsymbol{\tau}_{D}](\mathbf{x}) + \mathcal{S}\mathbf{f}_{ves} + \sum_{m=1}^{M} c_m\phi_m(\mathbf{x}).$$

So the full integral equation system that needs to be solve to find the unknowns $\boldsymbol{\tau}$ and $\mathbf{c}$ is

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}.$$

NOTE: Upon discretization, this system is not square and is not full rank.

Block solve

The solutions to

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

are given by

$$\mathbf{c} = -\mathbf{S}^{\dagger}(\mathbf{g} - \mathbf{C}\mathbf{A}^{-1}\mathbf{f})$$
$$\boldsymbol{\tau} = \mathbf{A}^{-1}\mathbf{f} - \mathbf{A}^{-1}\mathbf{B}\mathbf{c},$$

where $\mathbf{S} = \mathbf{Q} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is the matrix often referred to as the Schur complement.

Recall: $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_{-1} + \mathbf{A}_1$ where $\mathbf{A}_{-1} + \mathbf{A}_1$ is low rank.

The key for numerical approximations: Quadrature

Consider the integrable function $f(x)$ on an interval I. The set of points $\{x_j\}_{j=1}^N$ and weights $\{w_j\}_{j=1}^N$ satisfying

$$\int_I f(x)dx \sim \sum_{j=1}^{N} f(x_j)w_j$$

are called a quadrature rule.

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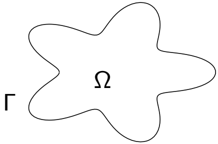
Quadrature for integral operators is challenging!

Th 4:00 - 4:45
Efficient and Accurate Discretization of Singular Integral Operators on Surfaces
James Bremer

F 10:30 - 11:15
On the solution of the biharmonic equation on regions with corners
Kirill Serkh

The particular solution



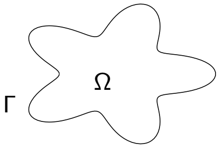
Recall, the particular solution is given by

$$v(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{y}) g(\mathbf{y}) dA(\mathbf{y}).$$

Applying an appropriate quadrature rule, the particular solution can be approximated by

$$v(\mathbf{x}) \sim \sum_{j=1}^{N} G(\mathbf{x}, \mathbf{y}_j) g(\mathbf{y}_j) w_j.$$

The particular solution



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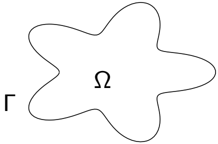
$$v(\mathbf{x}) \sim \sum_{j=1}^{N} G(\mathbf{x},\mathbf{y}_j)g(\mathbf{y}_j)w_j.$$

The evaluation of this sum can be accelerated with methods such as the FMM.

11:30 - 12:15
Adaptive grids for embedded integral equation based solvers
Travis Askham

The homogeneous solution



Consider the problem

$$\begin{aligned} -\Delta w(\mathbf{x}) &= 0, \qquad \mathbf{x} \in \Omega, \\ w(\mathbf{x}) &= \hat{f}(\mathbf{x}), \quad \mathbf{x} \in \Gamma. \end{aligned}$$

Recall the solution can be represented as a double layer potential

$$w(\mathbf{x}) = \int_\Gamma \partial_{\boldsymbol{\nu}_\mathbf{y}} G(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})ds(\mathbf{y}), \qquad \mathbf{x} \in \Omega,$$

where $\boldsymbol{\nu}_\mathbf{y}$ is the outward normal at $\mathbf{y}$ and $G(\mathbf{x},\mathbf{y})$ is the fundamental solution

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|.$$

Then the boundary charge distribution σ satisfies the boundary integral equation

$$\frac{1}{2}\sigma(\mathbf{x}) + \int_\Gamma \partial_{\boldsymbol{\nu}_\mathbf{y}} G(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})ds(\mathbf{y}) = \hat{f}(\mathbf{x})$$

How do you discretize integral equations?

To discretize the integral equation using a Nyström method, pick an appropriate quadrature to approximate the integral. Then

$$\begin{aligned}\hat{f}(\mathbf{x}) &= \frac{1}{2}\sigma(\mathbf{x}) + \int_\Gamma \partial_{\nu_{\mathbf{y}}} G(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})ds(\mathbf{y}) \\ &\sim \frac{1}{2}\sigma(\mathbf{x}) + \sum_{j=1}^{N} \partial_{\nu_{\mathbf{x}_j}} G(\mathbf{x},\mathbf{x}_j)\sigma(\mathbf{x}_j)w_j\end{aligned}$$

Looking for the solution at the quadrature nodes and forcing the approximation to hold at these locations leads to a linear system where the i^{th} row is given by

$$\hat{f}(\mathbf{x}_i) = \frac{1}{2}\sigma(\mathbf{x}_i) + \sum_{j=1}^{N} \partial_{\nu_{\mathbf{x}_j}} G(\mathbf{x}_i,\mathbf{x}_j)\sigma(\mathbf{x}_j)w_j$$

Model problem

Upon discretization, we have to solve a linear system of the form

$$\mathbf{A}\phi = (\frac{1}{2}\mathbf{I} + \mathbf{D})\boldsymbol{\sigma} = \hat{\mathbf{f}},$$

where $\mathbf{D}$ is a matrix that approximates the integral operator

$$\int_\Gamma \partial_{\nu_\mathbf{y}} G(\mathbf{x}, \mathbf{y})\sigma(\mathbf{y})ds(\mathbf{y}).$$

Properties of $\mathbf{A}$:

- Dense matrix.
- Size is determined by the number of discretization points.
- Data-sparse/structured matrix.

What does it mean for a matrix to be structured?

Roughly speaking, a matrix is structured if its off-diagonal blocks are low rank.

What do mean by low rank?

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What do mean by low rank?

Let $\mathbf{M}$ be an $m \times n$ matrix where $m \leq n$.

The Singular Value Decomposition (SVD) of $\mathbf{M}$ is a matrix factorization

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$$

where $\mathbf{U}$ and $\mathbf{V}$ are square unitary matrices and $\mathbf{\Sigma}$ is an $m \times n$ matrix with only positive real diagonal entries σ_j, $j = 1, \ldots, m$.

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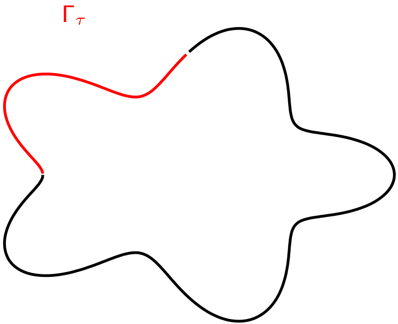
The values σ_j for $j = 1, \ldots, m$ are called the singular values.

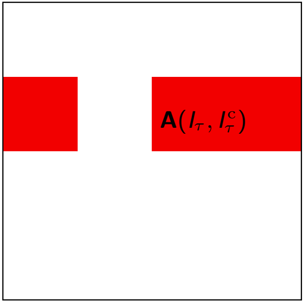
The ϵ-rank of a matrix is the number k of singular values greater than ϵ.

A matrix is numerically low rank if $k << m$.

Is the BIE data sparse?



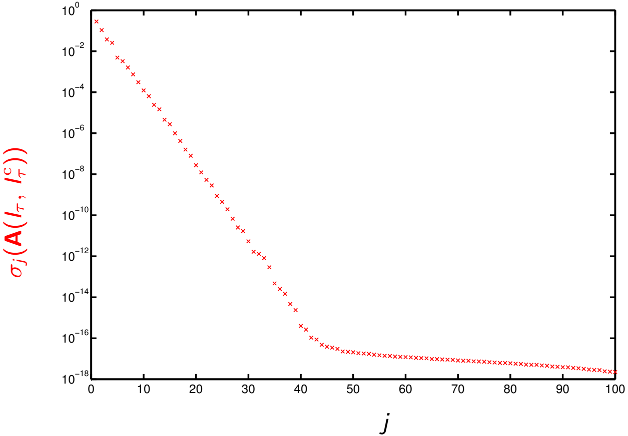
The contour Γ.



The matrix $\mathbf{A}$.

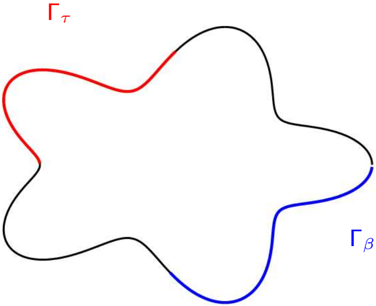
Is the BIE data sparse?

Singular values of $\mathbf{A}(I_\tau,\ I_\tau^{\rm c})$

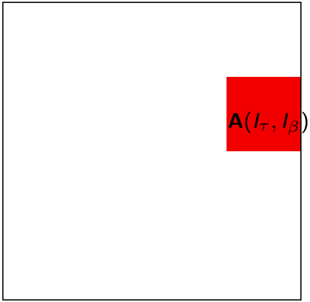


To precision 10^{-10}, the matrix $\mathbf{A}(I_\tau,\ I_\tau^{\rm c})$ has rank 29.

Is the BIE data sparse?



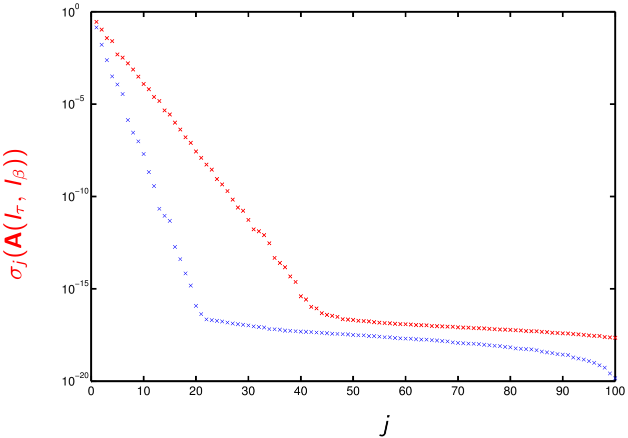
The contour Γ.



The matrix $\mathbf{A}$.

Is the BIE data sparse?

Singular values of $\mathbf{A}(I_\tau,\, I_\beta)$



To precision 10^{-10}, the matrix $\mathbf{A}(I_\tau,\, I_\beta)$ has rank 12.

Incomplete literature review — direct solvers based on data-sparsity:

1991 Data-sparse matrix algebra / wavelets, *Beylkin, Coifman, Rokhlin, et al*

1993 Fast inversion of 1D operators *V. Rokhlin and P. Starr*

1996 scattering problems, *E. Michielssen, A. Boag and W.C. Chew,*

1998 factorization of non-standard forms, *G. Beylkin, J. Dunn, D. Gines,*

1998 $\mathcal{H}$-matrix methods, *W. Hackbusch, B. Khoromskijet, S. Sauter, . . . ,*

2000 Cross approximation, matrix skeletons, etc., E. Tyrtyshnikov.

2002 $O(N^{3/2})$ inversion of Lippmann-Schwinger equations, *Y. Chen,*

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2002 (1999?) $\mathcal{H}^2$-matrix methods, *S. Börm, W. Hackbusch, B. Khoromskijet, S. Sauter.*

2004 Inversion of "FMM structure," *S. Chandrasekharan, T. Pals.*

2004 Proofs of compressibility, *M. Bebendorf, S. Börm, W. Hackbusch, ...*

2006 Accelerated nested diss. via $\mathcal{H}$-mats, *L. Grasedyck, R. Kriemann, S. LeBorne* [2007] *S. Chandrasekharan, M. Gu, X.S. Li, J. Xia.* [2010], *P. Schmitz and L. Ying.*

2010 construction of $\mathbf{A}^{-1}$ via randomized sampling, *L. Lin, J. Lu, L. Ying.*

Additional contributors: Ambikasaran, Bremer, Corona, Darve, Greengard, Ho, Martinsson, Michielssen, Rahimian, Zorin

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2010 construction of $\mathbf{A}^{-1}$ via randomized sampling, *L. Lin, J. Lu, L. Ying.*

Th 2:30 - 3:15

A fast direct solver for boundary value problems on locally perturbed geometries

Yabin Zhang

Scattering matrix: Trefoil

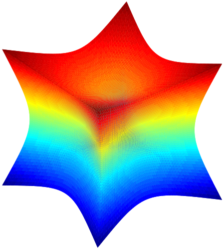
Consider a Dirichlet boundary value problem on the trefoil domain Ω with the wavenumber κ chosen so that the domain is approximately $1 \times 3 \times 3$ wavelengths in size. The tolerance is set to $\varepsilon = 1.0 \times 10^{-9}$. and 8th order quadrature is used.

N_{tris}	N	E_N	T	$N_{\text{out}} \times N_{\text{in}}$
16	832	6.73×10^{-04}	$1.17 \times 10^{+00}$	754×737
64	3 328	2.33×10^{-06}	$3.78 \times 10^{+01}$	939×910
256	13 312	2.59×10^{-08}	$3.61 \times 10^{+02}$	945×917
1 024	53 248	2.47×10^{-11}	$2.55 \times 10^{+03}$	948×918
4 096	212 992	-	$2.83 \times 10^{+04}$	949×921

Examples are from "A high-order accelerated direct solver for non-oscillatory integral equations on curved surfaces," with J. Bremer, and P.G. Martinsson.

Scattering Matrix: Corners and edges

Consider a Neumann boundary value problem on the deformed cube Ω with a fixed wavenumber $\kappa = \pi/2$ making the domain approximately 3.46 wavelengths diameter.



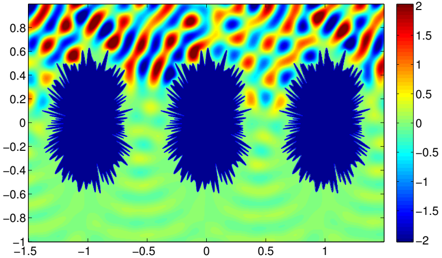
N_{tris}	N	E	T	$N_{\text{out}} \times N_{\text{in}}$
192	21 504	2.60×10^{-08}	$6.11 \times 10^{+02}$	617×712
432	48 384	2.13×10^{-09}	$1.65 \times 10^{+03}$	620×694
768	86 016	3.13×10^{-10}	$3.58 \times 10^{+03}$	612×685

$\varepsilon = 1.0 \times 10^{-10}$ 12th order quadrature

Examples are from "A high-order accelerated direct solver for non-oscillatory integral equations on curved surfaces," with J. Bremer, and P.G. Martinsson.

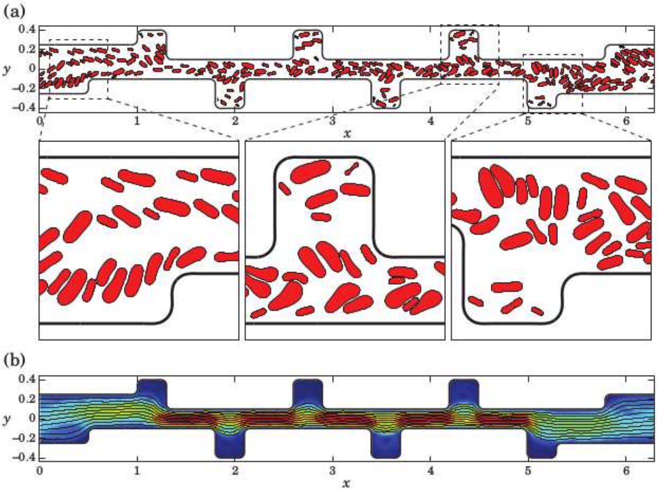
Free space scattering

Multiple incident waves



The FMM + GMRES takes one hour (248 iterations) to solve for the densities for one incident wave.
The fast direct solver takes 19.1 minutes to solve 200 densities.
(4.1 minutes of precomputation and 15 minutes for the block solves.)
Example from "A fast direct solver for quasi-periodic scattering problems," with A. Barnett, 2013.

Stokes flow



Example from "A fast algorithm for simulating multiphase flows through periodic geometries of arbitrary shape," with G. Marple, A. Barnett, and S. Veerapaneni, 2016.

Summary and concluding remark

Summary

- When a Green's function is available, constant coefficient PDEs can be reduced to solving a boundary integral equation and computing two convolutions.
- Variable coefficient PDEs can be recast as a volume integral equation.
- Well-conditioned integral equations may not be readily available but they can be designed.
- A variety of fast algorithms are available for both convolution and inversion.

Concluding remark

The field of fast algorithms for integral equations is young. Expect to see more exciting work from this community.